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# A survey of AOR and TOR methods

Jiaoxun KUANG and Jun JI
*Department of Mathematics, Shanghai Normal University, Shanghai, People's Republic of China*

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*Abstract:* This paper makes a survey on the AOR and TOR methods. We present the recent results about convergence and determination of the optimum parameters, mainly those obtained by Chinese authors.

*Keywords:* Iterative method, iteration matrix, convergence, convergence domain, optimum character.

## 1. Introduction

In his 1978 paper [1], Hadjidimos suggested a simple, but powerful scheme for large linear systems under the name of AOR method, which was defined by

$$x^{(n+1)} = \mathscr{L}_{\gamma,\omega} x^{(n)} + \omega (D - \gamma L)^{-1} b, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where

$$\mathscr{L}_{\gamma,\omega} = (D - \gamma L)^{-1} [(1-\omega) D + (\omega - \gamma) L + \omega U]$$

and $L + U = D - A$, $D = \mathrm{diag}(A)$, $L$ and $U$ are strictly lower and upper triangular matrices respectively.

Its simplicity compared with the other well-known methods (e.g. the SOR method) lies in the fact that for one complete AOR iteration, we need more than one matrix-vector multiplication, however, if conveniently stored, the amount of work involved for the computation of one complete AOR iteration is equivalent to that of an SOR iteration with the exception of the first iteration. Its powerfulness lies in the fact that two parameters instead of usually at most one, are present. As is pointed out in [1], full exploitation of the presence of these two parameters will provide us with methods which will converge faster than any other methods of the same type. Since the AOR method came into being, many numerical mathematicians have been and are still interested in it, and devote themselves to developing its theory. This paper is intended to give a survey on the recent developement of the AOR and the related GAOR and TOR methods in our country. It is impossible for us to include all the results obtained here in such a short note. Emphasis is put on the results about convergence domains and determinations of the optimum parameters.

## 2. Convergence theorems connected with matrix coefficients which possess some extra basic property

### *2.1. Irreducible matrices with weak diagonal dominance or strictly diagonally dominant matrices*

The well-known result for convergence for the special case was given by Hadjidimos in [1] which states: if $A$ is an irreducible matrix with weak diagonal dominance, then the AOR method converges for $0 \leqslant \gamma \leqslant 1$ and $0 < \omega \leqslant 1$. In [2] the authors established an extrapolation theorem which can be used to improve the above convergence domain. In [22] and [23], Martins got the upper bounds for the spectral radius of the AOR iteration matrix and improved the aforementioned result. In 1982, Martins [24] proved that if the matrix $A$ is irreducible weakly diagonally dominant or strictly diagonally dominant, then the AOR method is convergent for

$$0 \leqslant \gamma \leqslant p \quad \text{and} \quad 0 < \omega < \max\{p, g(\gamma)\}, \tag{2.1}$$

where $p \triangleq 2/(1 + \| \mathscr{L}_{0,1} \|_{\infty})$, $g(\gamma) \triangleq 2\gamma/(1 + \rho(\mathscr{L}_{\gamma,\gamma}))$ and $\rho(T)$ is the spectral radius of $T$.

The region determined by (2.1) is described in Fig. 1. Next, we give a result by Song Yongzhong. It is more general than the above-mentioned results.

**Theorem 1** [33]. *Let $A$ be a generalized diagonally dominant matrix, and $m(A)$ nonsingular, then $\rho(\mathscr{L}_{\gamma,\omega}(G)) < 1$ if*

$$0 \leqslant \gamma < 2/(1 + \rho(B)) \quad \textit{and} \quad 0 < \omega < \max\{2\gamma/(1 + \rho(\mathscr{L}_{\gamma,\gamma}(G))), 2/(1 + \rho(B))\} \tag{2.2}$$

*where the definitions of comparison matrix $m(A)$ and equimodular set of matrices, associated with $A$, $\Omega(A)$ can be found in* [40]. *$G \in \Omega(A)$, $\mathscr{L}_{\gamma,\omega}(G)$ is the iteration matrix of AOR method with $G$ instead of $A$ in* (1.1) *and $B \triangleq |\mathscr{L}_{0,1}(G)|$.*

Under the assumptions that $A$ is an irreducible matrix with weak diagonal dominance or a strictly diagonally dominant matrix, we have that $m(A)$ is nonsingular [40] and therefore the AOR method converges for

$$0 \leqslant \gamma < S \triangleq 2/(1 + \rho(\mathscr{L}_{0,1})) \quad \text{and} \quad 0 < \omega < \max\{2\gamma/(1 + \rho(\mathscr{L}_{\gamma,\gamma})), s\}, \tag{2.3}$$

see Fig. 2.

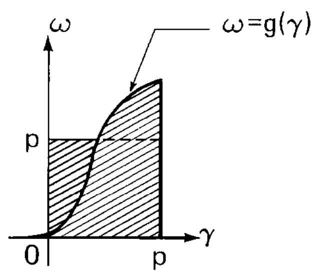

Fig. 1.

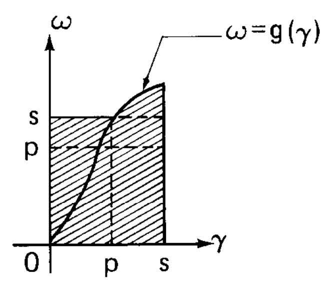

Fig. 2.

### 2.2. M- and H-matrices

The following is an immediate result of Theorem 1 because of the equivalence between H-matrices and generalized diagonal dominance (see [40]).

**Theorem 2.** *If $A$ is an H-matrix, then the AOR method is convergent for $\gamma$ and $\omega$ as given in* (2.3).

Obviously, Theorem 2 is an improvement of Theorem 6 in [24] and theorem 2 in [16]. Since an M-matrix must be an H-matrix by their definitions, we can see that Theorem 2 is also true for M-matrices.

### 2.3. Hermitian positive definite matrices

For this special case the AOR method (1.1) takes on the form:

$$x^{(n+1)} = (D-\gamma L)^{-1}\left[(1-\omega)D + (\omega-\gamma)L + \omega L^{\mathrm{H}}\right]x^{(n)} + \omega(D-\gamma L)^{-1}b, \tag{2.4}$$

where $D = \operatorname{diag}(A)$, $-L$ is the strictly lower triangular part of $A$ and $L^{\mathrm{H}}$ is the complex conjugate transpose of $L$. Many interesting results concerning (2.4) were obtained in [2], [27] and [12].

In [9] the authors considered a splitting of $A$ of Varga's type [41], i.e., $A = D - E - E^{\mathrm{H}}$ where $E = (D - A + S)/2$, $D$ is Hermitian and positive definite and $S$ is any skew-Hermitian matrix. Since $\det(D-\gamma E) \neq 0$ holds for all $\gamma \in [0, 2]$ [41], a generalization of the scheme (2.4) can be defined as:

$$x^{(n+1)} = (D-\gamma E)^{-1}\left[(1-\omega)D + (\omega-\gamma)E + \omega E^{\mathrm{H}}\right)x^{(n)} + \omega(D-\gamma E)^{-1}b \qquad n = 0, 1, 2, \dots \tag{2.5}$$

As for the scheme (2.5), in [9] (see also [28]) the authors gave the basic convergent result under the assumption that $\mu_m \leqslant 0 \leqslant \mu_M$ holds. $\mu_i$ $(i = 1, \dots, N)$ are the eigenvalues of the Jacobi matrix in (2.5), and $\mu_m = \min_i \mu_i$, $\mu_M = \max_i \mu_i$.

In [9], it is supposed that $\mu_m \leqslant 0 \leqslant \mu_M$ holds for the eigenvalues of the Jacobi matrix because only this case usually occurs in practice, where $D$ is either $\operatorname{diag}(A)$ or a block diagonal part of $A$. Theoretically, however, this does not cover the cases $\mu_m > 0$ or $\mu_M < 0$ with which the author dealt in [31]. In fact, a more generalized scheme defined below was studied in [31]

$$x^{(n+1)} = (D-\gamma E)^{-1}\left[(1-\omega)D + (\omega-\gamma)E + \omega E^{\mathrm{H}}\right]x^{(n)} + \omega(D-\gamma E)^{-1}b \qquad n = 0, 1, \dots, \tag{2.6}$$

where $D$ and $E$, which need not be diagonal and strictly lower triangular respectively as in (2.4), are just required to satisfy $E + E^{\mathrm{H}} = D - A$. It is also assumed that $\det(D-\gamma E) \neq 0$, so that the scheme (2.6) is well-defined.

**Theorem 3** [31]. *Let $A = D - E - E^{\mathrm{H}}$ be an Hermitian matrix and $D$ be Hermitian positive definite. If $(\gamma, \omega) \in J(\gamma, \omega)$ and* $\det(D-\gamma E) \neq 0$, *then the AOR method* (2.6) *converges iff $A$ is* *a Hermitian positive definite matrix where* $J = (\gamma, \omega) = \{(\gamma, \omega)\colon 0 < \omega < 2\}$ *satisfies the following conditions*:

(i) $\gamma > \omega + (2-\omega)/\mu_m$ *if* $\mu_m < \mu_M \leqslant 0$;

(ii) $\gamma < \omega + (2-\omega)/\mu_M$ *if* $\mu_M > \mu_m \geqslant 0$;

(iii) $\forall \gamma \in \mathbb{R}$ *if* $\mu_m = \mu_M = 0$;

(iv) $\omega + \dfrac{(2-\omega)}{\mu_m} < \gamma < \omega + \dfrac{(2-\omega)}{\mu_M}$ *if* $\mu_m < 0 < \mu_M$.

Obviously, Theorem 3 applies to the schemes (2.4) and (2.5), supplementing the results found in [28].

### 2.4. Consistently ordered matrices

In this section we assume that A is a consistently ordered matrix; that is, a matrix for which the expression $\det(\alpha L + \alpha^{-1}U - \beta D)$ is independent of $\alpha$ for $\alpha \neq 0$ and for all $\beta$.

#### 2.4.1. Convergence

The convergence of the AOR method for this case was first considered by Hadjidimos [1]. In [1] a necessary and sufficient condition was established when $\mathscr{L}_{0,1}$ has only real eigenvalues. According to this result, the convergence domain for $\gamma$ depends on the values of $\omega$. In [25] the authors approached the same problem in a new way and also got the necessary and sufficient convergence domains in the new form that the interval for $\omega$ depends on $\gamma$. Using the same method in [25], Missirlis described the necessary and sufficient convergence domains of the AOR method under the assumption that the Jacobi matrix possesses purely imaginary eigenvalues [26]. Although the same problem was also tackled independently in [35], the convergence domains in [35] were sufficient, not necessary. As a matter of fact, the convergence domains given in [25] and [26] extend those in [35].

#### 2.4.2. Optimum parameters

The first result about optimum parameters of the AOR goes back to 1978 and can be found in [1]. Under the assumption that the Jacobi matrix has only real eigenvalues $\mu_i$ $(i = 1, \ldots, N)$ such that $0 < \min_i |\mu_i| = \max_i |\mu_i| < 1$, Hadjidimos proved that $\rho(\mathscr{L}_{\gamma,\omega}) = 0$ for suitable parameters, which did ignite mathematicians' interest in the AOR method. In 1980, Tao Hua-Cheng proved the following results, which were published in 1984 [34].

**Theorem 4** [34]. *If* $A$ *is a consistently ordered 2-cycle matrix with nonvanishing diagonal elements such that Jacobi matrix possesses real eigenvalues* $\mu_i$ $(i = 1, \ldots, N)$ *and* $0 < |\mu_n| \leqslant \cdots \leqslant |\mu_2| \leqslant |\mu_1| < 1$, *then*

$$\inf_{\substack{\gamma \in (0,\infty) \\ \omega \in (0,\infty)}} \left\{ \rho\left(\mathscr{L}_{\gamma,w}\right) \right\} = \min_{\substack{\gamma \in [\gamma_n, \gamma_1] \\ \omega \in [\gamma_1, 2/\sqrt{1-\mu_n^2}]}} \rho\left(\mathscr{L}_{\gamma,w}\right)$$

$$= \min_{\omega \in [\gamma_1, 2/\sqrt{1-\mu_n^2}]} \left[ (\omega - 1)^2 + \omega \mu_n^2 (\gamma(\omega) - \omega) \right],$$

*where* $\gamma_1 = 2/(1 + \sqrt{1-\mu_1^2})$ *and* $\gamma_n = 2/(1 + \sqrt{1-\mu_n^2})$, ($\gamma(\omega)$ *see* [34]).

**Theorem 5** [34]. *Under the assumption of Theorem* 4, *we have*

$$\min_{\omega\in[\gamma_1,2/\sqrt{1-\mu_n^2}]} \rho\left(\mathscr{L}_{\gamma_1,\omega}\right)=\rho\left(\mathscr{L}_{\gamma_1,\gamma_1}\right)=\gamma_1-1 \quad \text{if } \mu_n^2\leqslant 1-\sqrt{1-\mu_1^2},$$

$$=\rho\left(\mathscr{L}_{\gamma_1,\omega_0}\right)=\sqrt{1-\frac{\left(2-\mu_n^2\gamma_1\right)^2}{4\left(1-\mu_n^2\right)}}<\gamma_1-1,$$

$$\text{if } \mu_n^2>1-\sqrt{1-\mu_1^2},$$

*where* $\omega_0=(2-\mu_n^2\gamma_1)/2(1-\mu_n^2)$.

If $0<|\mu_1|=|\mu_n|<1$, we have $\mu_n>1-\sqrt{1-\mu_1^2}$. By simple manipulation, we obtain from Theorem 5 that

$$\min_{\omega\in[\gamma_1,2/\sqrt{1-\mu_n^2}]} \left\{\rho\left(\mathscr{L}_{\gamma_1,\omega}\right)=0.\right.$$

So the result in [1] is an immediate corollary of theorems [4] and [5].

In [30] Sisler was able to find the optimum values for the cases:

(a) the Jacobi matrix has only real eigenvalues $\mu_i$ and $\min_i|\mu_i|=0$; or

(b) the Jacobi matrix has only purely imaginary eigenvalues $\pm\mu_i \mathrm{i}$ ($\mu_i$ real, $\mathrm{i}=\sqrt{-1}$) and $\min_i|\mu_i|=0$. However his proof was lengthy and complicated, and therefore a comparatively short and elementary one was given in [35]. In [14] the authors studied the same problem, provided that the eigenvalues of the Jacobi matrix are all real and less than 1 in modulus. But the detailed analysis, which was supplemented in [25], was not presented since "a tremendous number of cases" had to be examined. Using the method in [25], Missirlis [26] presented the optimum parameters provided that the eigenvaleus of the Jacobi matrix are all purely imaginary. By that time, the problem of optimum parameters concerning a 2-cycle matrix was almost solved.

## 3. Generalized AOR method

In [5] a method which generalizes the basic iterative methods for the solution of linear systems was proposed by A. Hadjidimos. According to [5] this new method has four degrees of freedom, which make it very flexible. What is more, this method is well-defined even when some elements on the diagonal of A are zero. [5] and [7] presented the first theoretical results. In [32] the author defined another generalized AOR method, which was first suggested by the authors in [28] in the case $A$ is Hermitian. According to [32], the GAOR method is defined by

$$x^{(n+1)}=\mathscr{L}_{\gamma,\omega}x^{(n)}+\omega(D-\gamma L)^{-1}b, \quad n=0,1,2,\ldots \tag{3.1}$$

and

$$\mathscr{L}_{\gamma,\omega}=(D-\gamma L)^{-1}[(1-\omega)D+(\omega-\gamma)L+\omega U].$$

In (3.1) $D$, $L$ and $U$, which need not be diagonal and strictly lower triangular and upper triangular respectively, are required to satisfy $A=D-L-U$. The new scheme is also well-defined even when some elements on the diagonal of $A$ are zero which is often the case in practice (for example see [11]). The example in [32] shows that a proper choice of $D$, $L$ and $U$ will make it a very powerful technique. Now, we give the first theoretical results of the GAOR scheme (3.1).

**Theorem 6** [32,36]. *If* $L \geqslant 0$, $U \geqslant 0$, $0 \leqslant \gamma$, $\omega \leqslant 1$, $\omega > 0$, *and* $\rho(\gamma L) < 1$, *then*
(a) $\rho(\mathscr{L}_{0,1}) = 0$ *iff* $\rho(\mathscr{L}_{\gamma,\omega}) = 1 - \omega$;
(b) $\rho(\mathscr{L}_{0,1}) = 1$ *iff* $\rho(\mathscr{L}_{\gamma,\omega}) = 1$;
(c) $0 < \rho(\mathscr{L}_{0,1}) < 1$ *iff* $1 - \omega < \rho(\mathscr{L}_{\gamma,\omega}) < 1$;
(d) $\rho(\mathscr{L}_{0,1}) > 1$ *iff* $\rho(\mathscr{L}_{\gamma,\omega}) > 1$.

**Theorem 7** [32]. *If* $L \geqslant 0$, $U \geqslant 0$, $(I - L - U)^{-1} \geqslant 0$, *and* $\rho(\gamma L) < 1$, *then* $\rho(\mathscr{L}_{\gamma,\omega}) < 1$ *for*

$$0 \leqslant \gamma < 2/(1 + \rho(\mathscr{L}_{0,1})) \quad \textit{and} \quad 0 < \omega < \max\left\{ \frac{2\gamma}{1 + \rho(\mathscr{L}_{\gamma,\gamma})}, \frac{2}{1 + \rho(\mathscr{L}_{0,1})} \right\}.$$

When $A$ is a Hermitian matrix, a convergent result similar to the Ostrowski–Reich theorem was given in [31] (see Theorem 3 of Section 2).

## 4. USAOR

In [38] Zhang Yin proposed a new method called USAOR which is defined by

$$\begin{cases} x^{(n+1/2)} = \mathscr{L}_{\gamma_1,\omega_1} x^{(n)} + \omega_1 (D - \gamma_1 L)^{-1} b, \\ x^{(n+1)} = \mathscr{U}_{\gamma_2,\omega_2} x^{(n+1/2)} + \omega_2 (D - \gamma_2 U)^{-1} b, \end{cases}$$

i.e.,

$$\begin{aligned} x^{(n+1)} = \mathscr{U}_{\gamma_2,\omega_2} \mathscr{L}_{\gamma_1,\omega_1} x^{(n)} + (D - \gamma_2 U)^{-1} [ (\omega_1 + \omega_2 - \omega_1\omega_2) D + \omega_2(\omega_1 - \gamma_1) L \\ + \omega_1 (\omega_2 - \gamma_2) U ] (D - \gamma_1 L)^{-1} b, \end{aligned} \tag{4.1}$$

where

$$\mathscr{L}_{\gamma_1,\omega_1} = (D - \gamma_1 L)^{-1} [ (1 - \omega_1) D + (\omega_1 - \gamma_1) L + \omega_1 U ],$$

$$\mathscr{U}_{\gamma_1,\omega_2} = (D - \gamma_2 U)^{-1} [ (1 - \omega_2) D + (\omega_2 - \gamma_2) U + \omega_2 L ].$$

It must be noted that for $\gamma_1 = \gamma$, $\omega_1 = \omega \neq 0$, $\gamma \neq \omega$, $\gamma_2 = \omega_2 = 0$ and $\gamma_1 = \gamma_2 = \gamma$, $\omega_1 = \omega_2 = \omega$, $\omega \neq \gamma$, we obtain the AOR and SAOR methods respectively.

In the special case that $A$ has the form:

$$A = \begin{pmatrix} I_1 & -H \\ -K & I_2 \end{pmatrix}, \tag{4.2}$$

where $I_1$ and $I_2$ are both identity matrices, a relationship between the eigenvalues $\lambda$ of the USAOR iterative matrix and the eigenvalues $\mu$ of the Jacobi matrix was established in [38]. This is the following:

$$\begin{aligned}
&[\lambda-(1-\omega_1)(1-\omega_2)]^2 \\
&=\{[\lambda-(1-\omega_1)(1-\omega_2)][\gamma_1\omega_1(1-\omega_2)+\gamma_2\omega_2(1-\omega_1)+\omega_1\omega_2(1+(1-\gamma_1)(1-\gamma_2))] \\
&\quad-\omega_1\omega_2(\omega_1-\gamma_1)(\omega_2-\gamma_2)\mu^2 \\
&\quad+[\omega_1(1-\omega_2)+\omega_2(1-\omega_1)(1-\gamma_2)][\omega_2(1-\omega_1)+\omega_1(1-\omega_2)(1-\gamma_1)]\}\mu^2.
\end{aligned}$$

In another paper [39], a comparison theorem concerning the SAOR and AOR methods was presented which can be formulated as follows:

**Theorem 8** [39]. *Let $A$ have the form* (4.2). *If $\mu^2<(4\gamma-1)/\gamma^2\leqslant 1$ or $((4\gamma-4)/\gamma^2+1)/2\leqslant\mu^2<1$ $\forall\gamma\neq 0$, $\omega$ and for $\mu\in\lambda(\mathscr{L}_{0,1})$, then $\rho^2(\mathscr{L}_{\gamma,\omega})\leqslant\rho(\mathscr{S}_{\gamma,\omega})$, where $\mathscr{L}_{\gamma,\omega}$ is the iteration matrix of the AOR method, $\mathscr{S}_{\gamma,\omega}$ is the iteration matrix of the SAOR method and $\lambda(T)$ is the spectrum of $T$.*

Next, we describe a convergent result from [38].

**Theorem 9** [38]. *If $A$ has the form of* (4.2) *and is symmetric positive definite, then the USAOR method* (4.1) *converges for*

$$0<\omega_i<2 \quad \text{and} \quad \omega_i-(2-\omega_i)/\rho<\gamma_i<\omega_i+(2-\omega_i)/\rho, \quad i=1,2,$$

*where $\rho=\rho(\mathscr{L}_{0,1})$ is the spectral radius of the Jacobi matrix.*

It is noted that it is hard for us to deal with the optimum parameters if four parameters are considered independently. In [38] Zhang Yin explored this nut under some restrictions, and gave two groups of parameters. Although they are not optimum, the analysis there shows that they are very effective.

## 5. TOR method

As is well-known, all the elements in $L$ and those in $U$ are considered as a whole in both AOR and SOR schemes. It does not seem reasonable since the case where the elements of $A$ vary a lot in magnitude usually occurs in practice. In [18] the lower part of $A$ is split into two parts: $E$ and $F$. Also, $E$ and $F$ are multiplied different factors, and this a new scheme was introduced.

Let $A=D-E-E^{\mathrm{H}}-F-F^{\mathrm{H}}$, where $-E$ and $-F$ are strictly lower triangular matrices. $D=\operatorname{diag}(A)$ is an Hermitian positive definite matrix. The TOR method is defined (according to [20]) by

$$x^{(n+1)}=\mathscr{L}_{\alpha,\beta,q}x^{(n)}+q(D-\alpha E-\beta F)^{-1}b, \quad n=0,1,2,\ldots, \tag{5.1}$$

where

$$\mathscr{L}_{\alpha,\beta,q}=(D-\alpha E-\beta F)^{-1}[(1-q)D+(q-\alpha)E+(q-\beta)F+q(E^{\mathrm{H}}+F^{\mathrm{H}})].$$

Obviously, if $\alpha = \beta$, the scheme (5.1) reduce to the AOR method; if we substitute $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ and $\frac{1}{2}(\alpha+\beta)$ for $\alpha$, $\beta$ and $q$ in (5.1) respectively, (5.1) becomes the TOR method given by the first author of this article.

$$x^{(n+1)} = (2D - \alpha E - \beta F)^{-1}\left[(2-\alpha-\beta)D + (\alpha+\beta)(E^{\mathrm{H}} + F^{\mathrm{H}}) + \beta E + \alpha F\right]x^{(n)} + (\alpha+\beta)(2D - \alpha E - \beta F)^{-1}b, \quad n = 0, 1, 2, \ldots . \tag{5.2}$$

A study of the scheme (5.2) was made in [18]. The following is the basic result.

**Theorem 10** [18,20]. *Let $A$ be Hermitian and positive definite. If $\tilde{D}$ is an Hermitian and positive definite matrix, then the scheme* (5.2) *converges for* $q > 0$, *while the scheme* (5.2) *diverges for* $q < 0$; *If $\tilde{D}$ is an Hermitian and negative definite matrix, the scheme* (5.2) *converges for* $q < 0$, *while the scheme* (5.2) *diverges for* $q > 0$, *where* $\tilde{D} = (2-q)D + (q-\alpha)(E + E^{\mathrm{H}}) + (q-\beta)(F + F^{\mathrm{H}})$.

Under the additional assumption that A is an L-matrix, some results were given in [18]. Further results for the cases where A is an H-matrix or irreducibly diagonally dominant were presented in [37]. Another special TOR method with $q = \alpha$ was discussed in [20].

In order that TOR method may be used to solve general linear systems (A may not be Hermitian) (5.1) was generalized in [17] and [21], to the following scheme:

$$x^{(n+1)} = (D - \alpha E - \beta F)^{-1}[(1-q)D + (q-\alpha)E + (q-\beta)F + qU]x^{(n)} + q(D - \alpha E - \beta F)^{-1}b, \quad n = 0, 1, 2, \ldots . \tag{5.3}$$

where $D = \mathrm{diag}(A)$; $-E$ and $-F$ are strictly lower triangular matrices and $U$ is strictly upper triangular matrix.

**Theorem 11** [21]. *Let $A = D - E - F - U$ be an H-matrix, then the scheme* (5.3) *converges for*

$$0 \leqslant \alpha < q < \frac{2}{1+\rho(B)} \quad \text{and} \quad 0 \leqslant \beta < q < \frac{2}{1+\rho(B)}$$

*where* $B \triangleq \mathscr{L}_{0,0,1}$.

Because of the equivalence between H-matrices and generalized diagonal dominance [40], the next theorem follows from Theorem 3 of [17].

**Theorem 12** [17]. *If $A$ is an H-matrix, then the scheme* (5.3) *converges for* $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$ *and* $0 < q \leqslant 1$.

**Theorem 13** [17]. *Let $D$ be nonsingular, and* $|\alpha|e_i+|\beta|l_i<1$, *then*

(i) $$\rho(\mathscr{L}_{\alpha,\beta,q})\leqslant\max_{1\leqslant i\leqslant N}\frac{|1-q|+|q-\alpha|e_i+|q-\beta|l_i+|q|f_i}{1-|\alpha|e_i-|\beta|l_i},$$

(ii) $$\rho(\mathscr{L}_{\alpha,\beta,q})\geqslant\min_{1\leqslant i\leqslant N}\frac{|1-q|-|q-\alpha|e_i-|q-\beta|l_i-|q|f_i}{1+|\alpha|e_i+|\beta|l_i},$$

*where* $e_i=\|D^{-1}E\|_\infty$, $l_i=\|D^{-1}F\|_\infty$ *and* $f_i=\|D^{-1}U\|_\infty$.

When $\alpha=\beta$, we obtain the bounds, which are also given by M. M. Martins in [22] and [23], for the spectral radius of the AOR method. However, it is easy to see that we establish these bounds without the assumption that $A$ is either strictly diagonally dominant or irreducibly diagonally dominant, which is needed in [22,23].

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